

# Some brane theoretic no-hair results (and their field theory duals)

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**Abstract.** This contribution to the proceedings of the 1999 Canadian Conference on General Relativity and Relativistic Astrophysics is a brief exposition of earlier work, with Sumati Surya (hep-th/9805121) Amanda Peet (hep-th/9903213), addressing certain results in higher dimensional supergravity that are related to black hole no-hair theorems. Its purpose is to describe, in language appropriate for an audience of relativists, how these results can be related to the Maldacena conjecture (aka, the AdS/CFT correspondence). The end product may be taken as a new kind of quantitative evidence in support of the Maldacena conjecture.

## I INTRODUCTION

The Maldacena conjecture [1] (aka the AdS/CFT correspondence) has been a topic of much interest and discussion in most communities with interests in quantum gravity. Here we provide some commentary on a few recent results which, in the end, provide a new type of quantitative check on this conjecture and its relatives [2], beyond those previously known (see [3] for a review). These new results relate to the effective ‘delocalization’ of charge near a black hole horizon, a phenomenon associated with black hole no-hair theorems. After discussing this phenomenon in the familiar context of 3+1 Einstein-Maxwell theory, we describe a related feature of ten-dimensional supergravity, which can then be related to the Maldacena conjecture. We will find that a corresponding phenomenon occurs in the so-called dual field theory, and that the supergravity and field theory results match at both the qualitative and quantitative levels.

Due to a shortage of space, both citations of the literature and inclusion of technical details will be minimal. In particular, as the intention is to make this paper accessible to newcomers to string theory and the Maldacena conjecture, stringy (and super Yang-Mills) details will be particularly sparse, and true string/field theorists are encouraged to go directly to the original works [4,5]. The goal of this paper is merely to provide a rough feel for the results and, perhaps, to motivate the reader to examine the original works.

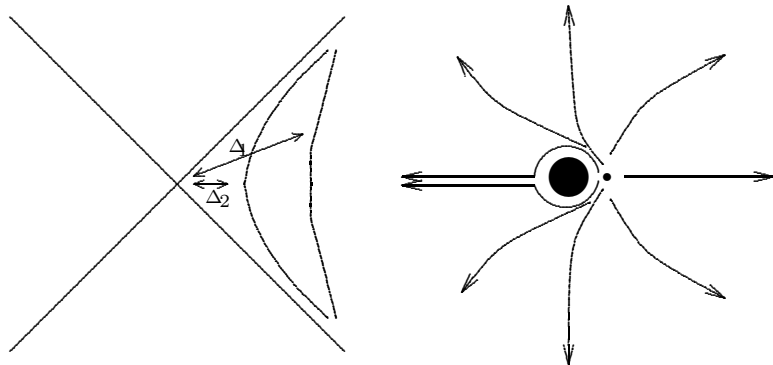
## II ON NO-HAIR RESULTS

Let us begin with a brief reminder of certain results associated with black hole no-hair theorems. For definiteness, consider Einstein-Maxwell theory (in 3+1 dimensions) in the presence of charged dust. Then we know that all stationary black hole solutions are parameterized by their mass  $M$ , their charge  $Q$ , and their angular momentum  $J$ . In particular, if  $J = 0$  then the solution is spherically symmetric.

Now suppose that we take a bit of charged dust and drop it into a black hole. The result will be a new black hole which will eventually settle down to a stationary state. In particular, if both the black hole and the bit of dust had  $J = 0$ , the result would be spherically symmetric. Thus, even though the bit of dust approaches from one side of the black hole, the electric field becomes spherically symmetric in the far future. As the charge approaches the black hole horizon, the electric field begins to become spherically symmetric due to the fact that the spacetime curvature will bend the electric field lines around the black hole. The result is that the electric charge appears (when viewed from far away) to be ‘spread out’ over the horizon of the black hole.

In fact, this effect is not sensitive to whether the electric charge actually falls through the horizon. Let us suppose that, at some point, the charge is attached to a powerful rocket that keeps it from falling further into the black hole, perhaps moving instead along one of the worldlines shown in the conformal diagram on the left below. The curvature of spacetime bends the electric field lines and tends to make the electric field spherically symmetric. A quasi-artistic impression of this process is shown in the diagram on the right below. There, the small dot denotes the bit of charge and the lines describe the electric field produced by the charge. The extent to which the electric field appears spherical is determined by just how close to the horizon the charge actually sits. In the limit in which the charge approaches the horizon, the electric field becomes spherically symmetric.

As a final conceptual jump, let us dispense with the rockets used above by passing to the extremal case, in which both the charged dust and the black hole satisfy  $Q = +M$ . Now, the electrostatic repulsion alone is sufficient to keep the charge from falling into the black hole. We could consider a family, labeled by a parameter  $\Delta$  as shown on the left below, of such extremal charges which do not fall toward the black hole at all. Instead, they sit close to an extremal hole following the integral curves of a timelike killing vector field. Once again, in the limit where the worldline of the charge approaches the horizon, the electric field becomes spherically symmetric and the charge appears to have spread out over the horizon of the black hole. We will refer to this effect as “delocalization” of the charge. Note that the diagram on the right below shows only the field lines produced by the additional charge, and not the field lines produced by the black hole itself.



We will shortly be interested in analogous phenomena in 10-d supergravity. What we will do is to use some qualitative features of this effect, along with certain aspects of the supergravity/field theory correspondence, to suggest an analogue in the field theory. Assuming this guess to be correct, the field theory makes both qualitative predictions about when effects of this sort should occur and also *quantitative* predictions<sup>1</sup> as to how fast the charge should delocalize as we adjust the parameter  $\Delta$ . Thus, it is worth thinking for a moment about how to build a quantitative measure of the delocalization in a black hole solution. A natural choice is to decompose the electric field using spherical harmonics, to measure the dipole, quadrupole, and higher moments of the electric field and to declare that the charge has delocalized on an angular scale  $\theta \sim 1/l$  when the charge is close enough to the black hole that the spherical harmonics of order  $l$  have become small. However, to do so, one must introduce a foliation of spacetime by spheres and an action of the rotation group on those spheres. Since, at finite  $\Delta$ , the spacetime is not spherically symmetric, there is some ambiguity here. A convenient choice is to use the spheres and  $SO(3)$  action defined by isotropic coordinates centered on the black hole. A calculation of the delocalization rate is then straightforward, as our spacetimes are described by the Majumdar-Papapetrou solutions [6], for which the fields take a simple form in isotropic coordinates. The detailed results are not important here, though they will be for the 10-d supergravity analogues discussed below.

### III THE SUPERGRAVITY VERSION

The main focus of this exposition is a version of charge delocalization in 9+1 dimensional supergravity theories. Recall that, in addition to black holes, higher dimensional supergravity has what are known as black *brane* solutions. A black brane is like a black hole except that its horizon does not have the topology of a sphere, and is often not compact. For example, in an  $n$  dimensional spacetime, an event horizon with topology  $S^{n-2} \times \mathbf{R}$ , where the  $\mathbf{R}$

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<sup>1</sup>) The term ‘prediction’ should perhaps be taken with a grain of salt. Even assuming both that the Maldacena conjecture is correct and that the field theory analogue of our effect has been correctly guessed, it is not a priori obvious that naive calculations of the type that we will discuss must be quantitatively correct. Nonetheless, that such naive results will in fact precisely agree with the supergravity seems rather impressive.

factor is associated with a null direction, would be referred to as a black hole, while an event horizon with topology  $S^{n-3} \times \mathbf{R} \times \mathbf{R}$  is a black string, and  $S^{n-2-p} \times \mathbf{R}^p \times \mathbf{R}$  is a black  $p$ -brane.

We will be particularly interested in the class of branes known as “D-branes” in the context of 9+1 supergravity theories. Here, the label D stands for ‘Dirichlet’ and D in no way denotes the dimension of the brane. Thus, we will often refer to  $Dp$ -branes; e.g., a D1-brane is a string of the “Dirichlet” type. For a discussion of what the term “Dirichlet” means in this context, see [7]. Below, we focus on the extremal limits of these solutions<sup>2</sup>. The associated supergravity solutions containing only a one brane have a single length scale,  $r_p \propto Q_p^{1/(7-p)}$ , where  $Q_p$  is the charge of the brane.

A relevant fact about  $Dp$ -branes is that branes of different dimensions (different values of  $p$ ) carry different kinds of charge. Thus, even in the extremal limit, one cannot in general construct static solutions containing  $Dp$ -branes with different values of  $p$ , as the gravitational attraction is not balanced by the electrostatic repulsion. However, there is also a dilaton field in the supergravity theories of interest, and this produces a repulsive force between two branes even when their dimensions differ. For the right combinations of extremal branes, it can be arranged for these dilatonic forces to hold the branes apart, so that exactly static solutions can in fact be constructed. A class of examples on which we will focus here consists of static spacetimes containing  $Dp$ -branes and  $D(p-4)$ -branes. Roughly speaking, static solutions exist whenever the  $D(p-4)$ -branes are ‘parallel’ to the  $Dp$ -branes. That is, the final solutions have a  $(p-4)$  dimensional set of commuting spacelike Killing vector fields<sup>3</sup>. We will focus on such systems from now on and, as a result, we can refer to the  $Dp$ -brane simply as ‘the big brane’ and the  $D(p-4)$ -brane simply as ‘the little brane’ or ‘the smaller brane.’ As before, we may consider a family of such solutions labeled by a parameter  $\Delta$  which describes the separation of the two branes.

Let us now think of the larger ( $p$ -dimensional) D-brane as the analogue of the black hole in section II, and let us think of the smaller ( $(p-4)$ -dimensional) brane as the analogue of the charge. We may again ask if the charge of the smaller brane ‘spreads out over the larger brane’ as the separation  $\Delta$  goes to zero. If it does not, then by taking the limit  $\Delta \rightarrow 0$ , one could form ‘hairy’ brane solutions which could not be completely characterized by the amount of big- and little-brane charge they carried. Instead, they would also require the specification of the *distribution* of the little-brane charge over the big brane.

Now, it is not a priori clear what our 3+1 Einstein-Maxwell intuition should tell us about the delocalization of little-brane charge in the current context. We now consider to a different gravitating theory and, while the objects being described are similar to black holes, they are singular. It turns out that it is again possible to study delocalization by direct calculation as the analogues of the Majumdar-Papapetrou metrics can be found for an arbitrary separation  $\Delta$  between the branes. Such solutions were studied in [4] for the case of  $p=5$ , and more generally in [5]. The final result is that the little-brane delocalizes for the cases  $p=4$  and  $p=5$ , but not for  $p=6$ . That is, for  $p=4,5$  the limiting soliton as  $\Delta \rightarrow 0$  in fact has a  $p$ -dimensional group of translation symmetries in addition to the rotational symmetries that one would expect of a  $p$ -brane. However, for  $p=6$ , the only new symmetries of the limiting solution are the rotational ones. Corresponding localized solutions were in fact constructed explicitly in [8] in what is known as the “near-core limit.” One might say that, for  $p=6$ , the 2-brane charge spreads out in the angular directions around the 6-brane, but not in the translational directions along the 6-brane. It is probably not a coincidence that D4-branes and D5-branes have (naked) null singularities, while the D6-brane in fact has a naked timelike singularity. Following a common practice, we will not discuss cases with  $p \geq 7$ . The reason for this is that such solutions have a more complicated structure at infinity. Since our branes live in a (9+1)-dimensional spacetime,  $p=7$  branes behave like point particles in 2+1 dimensions, producing conical deficit angles at infinity. The  $p=8$  branes produce fields which do not fall off at infinity, and the  $p=9$  branes extend to infinity in all directions.

One can measure the rate at which charge delocalizes in a manner similar to that discussed in section II. Here, we are most interested in the delocalization in the directions along the brane. Thus, we can simplify our lives by considering not just a single (fully localized) little brane, but a shell of such branes placed in a spherically symmetric manner about the bigger-brane. Thus, even at finite  $\Delta$ , we may consider solutions that have the symmetries of  $S^{8-p}$  in addition to a  $(p-4)$ -dimensional translational symmetry.

These spacetimes may be equipped with a nice set of coordinates [5] analogous to the isotropic coordinates of the Majumdar-Papapetrou solutions. This introduces a radial function  $r$  and an action of the four-dimensional translation group that moves the little brane along the big-brane. To measure the extent to which little-brane charge is delocalized, consider some surface  $r=r_0$  outside the shell of little branes and Fourier transform the fields on this surface with respect to the four-dimensional translation group. For a shell at  $r=\Delta < r_0$ , we may say that the solution has delocalized on a length scale  $\lambda$  when the corresponding Fourier component falls below, say,  $e^{-10}$  times the value it has when the shell is at  $r=r_0$ . In fact, we may take a limit  $r_0 \rightarrow \infty$  to make this definition

<sup>2)</sup> Unfortunately, the extremal limits. Instead, they are singular static solutions, with the norm of the Killing field vanishing at the singularity. The singularity is null for  $p \leq 5$  but timelike for  $p \geq 6$ .

<sup>3)</sup> In fact, the symmetry group contains the Poincare’ group of  $(p-4)+1$  dimensional Minkowski space.

independent of  $r_0$ . We display the results here so that the reader may properly appreciate the comparison with the field theoretic calculations in section IV. In the limit where the separation  $\Delta$  between the branes is small, the distance scale over which the solution is delocalized behaves like  $\lambda \sim r_4^{3/2}/\Delta^{1/2}$  for the case  $p = 4$ , and it behaves like  $\lambda \sim r_5 \sqrt{\ln(r_5/\Delta)}$  for  $p = 5$ , while the  $p = 6$  case does not delocalize.

After reading this section, the traditional relativist may feel caught up in a swirl of D's,  $p$ 's,  $Q$ 's and various other letters of the alphabet. Such a reader should take a moment to collect their thoughts, as this final section will not be more familiar. In regard to many statements below, the traditional relativist may feel unqualified to just what is 'reasonable' or 'natural.' This is, of course, to be expected with a new subject, and I urge such a reader not to worry overly much. The main goal should be to take away a broad overview of the argument and some appreciation for the results.

## IV FIELD THEORY DUALS

Now, describing charge delocalization in supergravity is all well and good, but the current excitement in string theory concerns the so-called Maldacena conjecture. This conjecture states that aspects of supergravity are described by certain quantum field theories, even though those theories do not include gravity when viewed in the usual way. For details, see the original papers [1,2] or the recent review [3]. Here we content ourselves with an extremely rough statement of the conjecture, which is that supergravity physics near the 'horizon' (the locus where the norm of the static Killing field vanishes) of a  $Dp$ -brane is in fact completely described by a (non-gravitating) quantum field theory. The idea is that simple supergravity quantities supergravity may in principle be quite complicated when written in terms of the gauge theory, but that nevertheless such a 'dictionary' that tells us how to translate supergravity physics into field theory physics can in fact be constructed. Furthermore, this dictionary has the property that classical gravitational effects on the supergravity side of this correspondence are mapped to strongly quantum mechanical effects on the field theory side. For a discussion of how our current context of multiple separated branes fits into the Maldacena conjecture, see [5].

For now, we simply quote a few results that will paint a backdrop for the connections we wish to make between supergravity and field theory. The fact that we are interested in the near-horizon physics of the  $Dp$ -brane means that we will be considering  $SU(N)$  super Yang-Mills theory in  $p + 1$  dimensions. Here, the value of  $N$  is related to the charge of the branes, the string length  $l_s$ , and the string coupling  $g_s$  through  $r_p = l_s(g_s N)^{1/(7-p)}$ . Now, it is well known that Yang-Mills theory in 4 Euclidean dimensions contains instantons. As a result, Yang-Mills theory in  $p + 1$  dimensions (for  $p \geq 4$ ) contains solitons which are just the lift of the instanton solutions to  $p + 1$  dimensions. The result is a  $(p - 4)$ -brane shaped soliton, and such solitons in the Yang-Mills theory are to be associated in the Maldacena conjecture with the  $D(p - 4)$ -branes of the supergravity theory.

The next part of our story will be to guess what aspect of the Yang-Mills theory should correspond to the spreading of the charge in the supergravity solution. This is certainly a guess, as it is not something which can be derived from the known aspects of the correspondence. However, we will see that there is an extremely natural candidate. Now, a soliton can be viewed as a coherent 'lump' of classical field that holds itself together through the non-linear dynamics of the field. It turns out that, in super Yang-Mills theory, the size  $\rho$  of this lump has no preferred value. Taking any instanton solution to the equations of motion and scaling it by a constant factor again gives a static solution. Thus, there are static soliton solutions in our  $p + 1$  Yang-Mills theory with any value of the scale size  $\rho$ .

Furthermore, one may allow  $\rho$  to vary over the  $(p - 4) + 1$  dimensional world-volume of the soliton. Such solutions are not static, but if the distortions are small and of long wavelength (much larger than the string length) it turns out that  $\rho(x)$  behaves just like a massless  $(p - 4) + 1$  dimensional field. To be a bit more precise, because the soliton can point in roughly  $N$  directions in gauge space,  $\rho$  can be thought of as a roughly  $N$ -dimensional vector of massless scalars. Thus,  $\rho^2$  acts like a sum  $\sum_{i=1}^N \phi_i^2$  over massless scalar fields. Normalizing the scalar fields  $\phi_i$  canonically, one finds  $\rho^2 = l_s^{(p-3)} g_s \sum_{i=1}^N \phi_i^2$ .

The soliton in the Yang-Mills theory gives a nice picture of a  $D(p - 4)$ -brane sitting inside a  $Dp$ -brane, but one might ask how to encode a separation of the two branes into the Yang-Mills theory. This is done through another set of fields and, for our purposes, the important property is that, when these fields take some nonzero value  $\Delta$ , a mass scale  $m(\Delta)$  is generated which interacts with the field  $\rho$ . If the  $D(p - 4)$ -brane shell is located at  $r = \Delta$ , then this mass is known to be  $m = l_s^{-2} \Delta$ , in units where  $\hbar = 1$ . This means that our description of  $\rho$  as a free field is really only valid for wavelengths shorter than some infra-red cutoff  $\Lambda_{IR}(\Delta) = l_s^2/\Delta$  and for wavelengths longer than the string scale due to the short-distance cutoff described above.

So, since  $\rho$  tells us how spread out the instanton is in the appropriate directions, it is natural to expect that it corresponds, via the Maldacena conjecture, to the spread of the  $D(p - 4)$ -brane charge in supergravity, at least when the  $D(p - 4)$  brane is close to the horizon of the  $Dp$ -brane. Now, in order to match the energy of the extremal supergravity solution, the energy in the field theory must be that of a ground state. Let us therefore ask how large

$\rho$  should be in a typical low-energy quantum state of the gauge theory. Classically,  $\rho$  can be set to any size we desire without changing the energy. Quantum mechanically, an attempt to confine  $\rho$  to a small range of values, minimizing the uncertainty in  $\rho$ , requires a large momentum conjugate to  $\rho$  and thus a large energy. Thus, quantum fluctuations will effectively cause the value of  $\rho$  to be non-zero in the ground state. We can estimate a scale for the effective size of  $\rho$  by computing, for example,  $\langle \rho^2 \rangle$  for the various cases. To do so, we use the fact that, between the appropriate infra-red and ultra-violet cutoff scales,  $\rho$  acts much like a free field. Outside this range of wavelengths, it is reasonable to assume that the fluctuations of  $\rho$  are small. Thus, we can estimate  $\langle \rho^2 \rangle$  using the properties of massless scalar fields in various dimensions. The result is

$$\begin{aligned} \text{For } p = 4, \quad \langle \rho^2 \rangle &= (l_s g_s) N (\Lambda_{IR} - \Lambda_{UV}), \\ \text{For } p = 5, \quad \langle \rho^2 \rangle &= (l_s^2 g_s) N \ln(\Lambda_{IR}/\Lambda_{UV}), \\ \text{For } p = 6, \quad \langle \rho^2 \rangle &= (l_s^3 g_s) N (\Lambda_{UV}^{-1} - \Lambda_{IR}^{-1}). \end{aligned} \tag{1}$$

Converting to the supergravity parameters and using the near-horizon limit, these results become

$$\begin{aligned} \text{For } p = 4, \quad \sqrt{\langle \rho^2 \rangle} &\sim r_4^{3/2} \Delta^{-1/2}, \\ \text{For } p = 5, \quad \sqrt{\langle \rho^2 \rangle} &\sim r_5 \sqrt{\ln(\Delta/r_5)}, \\ \text{For } p = 6, \quad \sqrt{\langle \rho^2 \rangle} &\sim r_6^{1/2} l_s^{1/2}. \end{aligned} \tag{2}$$

Note that for  $p = 4$  and  $p = 5$  these agree precisely with the delocalization rates of the supergravity solutions, while for  $p = 6$  the field theory predicts that the delocalization is bounded as  $\Delta \rightarrow 0$  by a value proportional to a positive power of the string scale; i.e., that it is small compared to the scales of classical supergravity. As explained in [5], similar arguments in the field theory tell us that certain other kinds of charge delocalization should also occur in supergravity. Due to their more complicated nature, these predictions are harder to check directly. Nevertheless, some preliminary investigations [9] seem to support these predictions. Thus, the phenomenon of charge delocalization provides a new kind of evidence in support of the Maldacena conjecture. We hope that it may also provide new insight into the nature of the supergravity/field theory correspondence.

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